

# Comment on "Approximate solutions of the Dirac equation for the Rosen-Morse potential including the spin-orbit centrifugal term"

A. Ghoumaïd, F. Benamira and L. Guechi  
Laboratoire de Physique Théorique, Département de Physique,  
Faculté des Sciences Exactes, Université des Frères Mentouri, Constantine,  
Route d'Ain El Bey, Constantine, Algeria

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## Abstract

It is shown that the application of the Nikiforov-Uvarov method by Ikhdair for solving the Dirac equation with the radial Rosen-Morse potential plus the spin-orbit centrifugal term is inadequate because the required conditions are not satisfied. The energy spectra given are incorrect and the wave functions are not physically acceptable. We clarify the problem and prove that the spinor wave functions are expressed in terms of the generalized hypergeometric functions  ${}_2F_1(a, b, c; z)$ . The energy eigenvalues for the bound states are given by the solution of a transcendental equation involving the hypergeometric function.

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Recent paper [1] paid attention to the bound state problem of the Dirac equation for the radial Rosen-Morse potential plus the spin-orbit centrifugal term under the conditions of the spin and pseudospin symmetries by means of the Nikiforov-Uvarov method (NU) [2]. This problem has been reduced to the solution of the respective Schrödinger-like bound state equations

$$\begin{aligned} & \left[ -\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + \frac{1}{\hbar^2 c^2} (Mc^2 + E_{n_r, \kappa} - C_s) \Sigma(r) \right] F_{n_r, \kappa}(r) \\ &= \frac{1}{\hbar^2 c^2} [E_{n_r, \kappa}^2 - M^2 c^4 + C_s (Mc^2 - E_{n_r, \kappa})] F_{n_r, \kappa}(r); \quad F_{n_r, \kappa}(0) = F_{n_r, \kappa}(\infty) = 0, \end{aligned} \quad (1)$$

(see eq.(20) in Ref.[1]) and

$$\begin{aligned}
& \left[ -\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2} - \frac{1}{\hbar^2 c^2} (Mc^2 - E_{n_r, \kappa} + C_{ps}) \Delta(r) \right] G_{n_r, \kappa}(r) \\
&= \frac{1}{\hbar^2 c^2} [E_{n_r, \kappa}^2 - M^2 c^4 - C_{ps} (Mc^2 + E_{n_r, \kappa})] G_{n_r, \kappa}(r); \quad G_{n_r, \kappa}(0) = G_{n_r, \kappa}(\infty) = 0,
\end{aligned} \tag{2}$$

(see eq. (22) in Ref.[1]) where  $M$  is the mass of the particle,  $E_{n_r, \kappa}$  is the bound state energy,  $\kappa$  denotes the spin-orbit quantum number,  $C_s$  and  $C_{ps}$  are two constants and the functions  $\Sigma(r)$  and  $\Delta(r)$  are identical and represent a certain phenomenological external potential which has been taken in the Rosen-Morse form

$$\Sigma(r) = \Delta(r) = -4V_1 \frac{\exp(-2\alpha r)}{(1 + \exp(-2\alpha r))^2} + V_2 \frac{1 - \exp(-2\alpha r)}{1 + \exp(-2\alpha r)}. \tag{3}$$

The use of the following approximation

$$\frac{1}{r^2} \approx \frac{1}{r_e^2} \left[ D_0 - D_1 \frac{\exp(-2\alpha r)}{1 + \exp(-2\alpha r)} + D_2 \left( \frac{\exp(-2\alpha r)}{1 + \exp(-2\alpha r)} \right)^2 \right], \tag{4}$$

and the elementary change of variables

$$z = -\exp(-2\alpha r) \tag{5}$$

reduce eqs. (1) and (2) to the Gauss's hypergeometric differential equations

$$\left[ z(1-z) \frac{d^2}{dz^2} + (1-z) \frac{d}{dz} - \frac{\beta_1 z^2 - \beta_2 z + \varepsilon_{n_r, \kappa}^2}{z(1-z)} \right] F_{n_r, \kappa}(z) = 0; \quad F_{n_r, \kappa}(-1) = F_{n_r, \kappa}(0) = 0, \tag{6}$$

$$\left[ z(1-z) \frac{d^2}{dz^2} + (1-z) \frac{d}{dz} - \frac{\bar{\beta}_1 z^2 - \bar{\beta}_2 z + \bar{\varepsilon}_{n_r, \kappa}^2}{z(1-z)} \right] G_{n_r, \kappa}(z) = 0; \quad G_{n_r, \kappa}(-1) = G_{n_r, \kappa}(0) = 0, \tag{7}$$

where the quantities  $\varepsilon_{n_r, \kappa}$ ,  $\beta_1, \beta_2, \bar{\varepsilon}_{n_r, \kappa}, \bar{\beta}_1$  and  $\bar{\beta}_2$  are defined by eqs. (28a), (28b), (28c), (45a), (45b) and (45c) in Ref.[1], respectively. Note that a minus sign is missing in the boundary conditions (see eqs. (27) and (44) of Ref.[1]).

At this stage, unfortunately, the author of the Ref. [1] employs the polynomial method (NU) to solve the differential equations (6) and (7). To implement this approach, he gave to expressions of the polynomial  $\sigma(z)$  and of the functions  $\rho(z)$  and  $\phi(z)$  the following canonical forms (see eqs. (29), (37) and (38) in Ref.[1]):

$$\sigma(z) = z(1-z), \tag{8}$$

$$\rho(z) = z^{2\varepsilon_{n_r, \kappa}}(1-z)^{2\delta+1}, \quad (9)$$

and

$$\phi(z) = z^{\varepsilon_{n_r, \kappa}}(1-z)^{\delta+1}. \quad (10)$$

for the spin symmetry solution and just replace  $\varepsilon_{n_r, \kappa}$  and  $\delta$  by  $\bar{\varepsilon}_{n_r, \kappa}$  and  $\delta_1$  for the pseudospin symmetry solution. Then, he claimed to have obtained the solutions of eqs. (6) and (7) as

$$\begin{aligned} F_{n_r, \kappa}(z) &= \phi(z) y_{n_r}(z) \\ &= \mathcal{N}_{n_r, \kappa} z^{\varepsilon_{n_r, \kappa}} (1-z)^{\delta+1} P_{n_r}^{(2\varepsilon_{n_r, \kappa}, 2\delta+1)}(1-2z), \end{aligned} \quad (11)$$

and

$$\begin{aligned} G_{n_r, \kappa}(z) &= \phi(z) y_{n_r}(z) \\ &= \mathcal{N}_{n_r, \kappa} z^{\bar{\varepsilon}_{n_r, \kappa}} (1-z)^{\delta_1+1} P_{n_r}^{(2\bar{\varepsilon}_{n_r, \kappa}, 2\delta_1+1)}(1-2z), \end{aligned} \quad (12)$$

where  $P_{n_r}^{(\mu, \nu)}(x)$  is the Jacobi polynomial.

However, there are some serious points that invalidate these solutions. First, they do not satisfy the boundary conditions  $F_{n_r, \kappa}(-1) = 0$  and  $G_{n_r, \kappa}(-1) = 0$ . So they are not physically acceptable. Next, it should be noted that the weight  $\rho(z)$  does not satisfy the condition

$$\sigma(z)\rho(z)z^k \Big|_a^b = 0, \quad (k = 0, 1, \dots), \quad (13)$$

(see theorem on the orthogonality of polynomials of hypergeometric-type in Ref. [2], eq. (17), p. 29). Here  $(a, b) \equiv (-1, 0)$ . Then, the hypergeometric functions  $y_{n_r}(z)$  are not orthogonal polynomials on the interval  $(-1, 0)$  and in this case, one can not extract the eigenvalues of the energy from the equation (see Ref. [2], eq. (13), p. 9),

$$\lambda_{n_r} + n_r \tau' + \frac{n_r(n_r - 1)}{2} \sigma'' = 0; \quad (n_r = 0, 1, \dots). \quad (14)$$

Therefore, we must discard the above solutions entirely and the energy equations given by (34) and (47) in Ref. [1] are not correct.

Since the polynomial method (NU) cannot be applied to the discussion of the radial Rosen-Morse problem, we otherwise proceed to determine the energy spectrum and the corresponding wave functions. In the case of spin symmetry, we look for the solution of the equation (6) in the form

$$F_{n_r, \kappa}(z) = z^\mu (1-z)^\delta y_{n_r}(z), \quad (15)$$

in which, on account of boundary conditions,  $\mu$  has to be positive and  $\delta$  may be a real quantity. Substituting (15) into (6) and taking

$$\mu = \varepsilon_{n_r, \kappa}, \quad (16)$$

and

$$\delta_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \beta_1 - \beta_2 + \varepsilon_{n_r, \kappa}^2}, \quad (17)$$

we obtain for  $y_{n_r}(z)$  the differential equation

$$\left\{ z(1-z) \frac{d^2}{dz^2} + [2\mu + 1 - (2\mu + 2\delta + 1)z] \frac{d}{dz} - (\mu + \delta)^2 + \beta_1 \right\} y_{n_r}(z) = 0. \quad (18)$$

The solution of this equation is the hypergeometric function

$$y_{n_r}(z) = \mathcal{N} {}_2F_1 \left( \mu + \delta_{\pm} + \sqrt{\beta_1}, \mu + \delta_{\pm} - \sqrt{\beta_1}, 2\mu + 1, z \right). \quad (19)$$

Now, taking into account the formulas (see Ref. [3], eqs. (9.131), p. 1043),

$${}_2F_1(\alpha, \beta, \gamma; z) = (1-z)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta, \gamma; z), \quad (20)$$

$${}_2F_1(\alpha, \beta, \gamma; z) = (1-z)^{-\alpha} {}_2F_1 \left( \alpha, \gamma - \beta, \gamma; \frac{z}{z-1} \right), \quad (21)$$

the upper component of the radial spinor wave function has the following expression

$$\begin{aligned} F_{n_r, \kappa}(r) &= \mathcal{N} \left( -e^{-2\alpha r} \right)^{\varepsilon_{n_r, \kappa}} (1 + e^{-2\alpha r})^{-\varepsilon_{n_r, \kappa} - \sqrt{\beta_1}} \\ &\times {}_2F_1 \left( \varepsilon_{n_r, \kappa} + \delta_+ + \sqrt{\beta_1}, \varepsilon_{n_r, \kappa} - \delta_+ + \sqrt{\beta_1} + 1, 2\varepsilon_{n_r, \kappa} + 1, \frac{1}{e^{2\alpha r} + 1} \right), \end{aligned} \quad (22)$$

where  $\mathcal{N}$  is a constant factor. The solution (22) fulfills the boundary condition  $F_{n_r, \kappa}(0) = 0$ , when

$${}_2F_1 \left( \varepsilon_{n_r, \kappa} + \delta_+ + \sqrt{\beta_1}, \varepsilon_{n_r, \kappa} - \delta_+ + \sqrt{\beta_1} + 1, 2\varepsilon_{n_r, \kappa} + 1, \frac{1}{2} \right) = 0. \quad (23)$$

Then, the energy values for the bound states are given by the solution of this transcendental equation (23) which can be solved numerically.

In order to obtain the pseudosymmetry solution, we proceed similarly as for the previous case and obtain the lower component of radial spinor wave function

$$\begin{aligned} G_{n_r, \kappa}(r) &= \overline{\mathcal{N}} \left( -e^{-2\alpha r} \right)^{\overline{\varepsilon}_{n_r, \kappa}} (1 + e^{-2\alpha r})^{-\overline{\varepsilon}_{n_r, \kappa} - \sqrt{\beta_1}} \\ &\times {}_2F_1 \left( \overline{\varepsilon}_{n_r, \kappa} + \overline{\delta}_+ + \sqrt{\beta_1}, \overline{\varepsilon}_{n_r, \kappa} - \overline{\delta}_+ + \sqrt{\beta_1} + 1, 2\overline{\varepsilon}_{n_r, \kappa} + 1, \frac{1}{e^{2\alpha r} + 1} \right), \end{aligned} \quad (24)$$

where

$$\overline{\delta}_+ = \frac{1}{2} + \sqrt{\frac{1}{4} + \overline{\beta}_1 - \overline{\beta}_2 + \overline{\varepsilon}_{n_r, \kappa}^2}, \quad (25)$$

and  $\overline{\mathcal{N}}$  is a constant factor. Then, the energy spectrum can be also found from a numerical solution of the transcendental equation

$${}_2F_1\left(\overline{\varepsilon}_{n_r,\kappa} + \overline{\delta}_+ + \sqrt{\overline{\beta}_1}, \overline{\varepsilon}_{n_r,\kappa} - \overline{\delta}_+ + \sqrt{\overline{\beta}_1} + 1, 2\overline{\varepsilon}_{n_r,\kappa} + 1, \frac{1}{2}\right) = 0. \quad (26)$$

In conclusion, given boundary conditions, the problem of solving the equations (6) and (7) does not belong to the class of quantum mechanical problems when it comes to search for energy levels and wave functions of a quantum system moving in a field of forces using classical orthogonal polynomials as eigenfunctions.

## References

- [1] S. M. Ikhdair, J. Math. Phys. 51, 023525 (2010)
- [2] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics* (Birkhäuser, Basel, 1988).
- [3] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series and products* (Academic Press, New York, 1965).